Many More Predecessors: A Representation Workout

Oleg Kiselyov
http://okmij.org/ftp/Computation/lambda-calc.html#predecessors

Tohoku University, Japan

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Outline

Subject

Pure untyped lambda-calculus and Church numerals

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\mathsf{c}_0 &:= \quad \lambda f.\lambda x.x \\
\mathsf{c}_1 &:= \quad \lambda f.\lambda x.f\ x \\
\mathsf{c}_2 &:= \quad \lambda f.\lambda x.f\ (f\ x) \\
&\quad \ldots \\
\mathsf{c}_n &:= \quad \lambda f.\lambda x.f^{(n)}x
\end{aligned}
$$

Successor	$\mathsf{succ}\ \mathsf{c}_n \rightsquigarrow^* \mathsf{c}_{(n+1)}$
	$\mathsf{succ} := \lambda n.\lambda f x.f\ (n\ f\ x)$
Predecessor	$\mathsf{pred}\ \mathsf{c}_{(n+1)} \rightsquigarrow^* \mathsf{c}_n$

Motivation

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Why did I care?

Why did I care, initially

Pure untyped lambda-calculus and Church numerals

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$\mathsf{pred} := \lambda n.n\ (\lambda p.p\ (\lambda x.x)(\lambda f x.f\ (p\ f\ x)))\ (\lambda f x s z.z)$

(two days + tooth)

Summary

It is a representation-change problem

Results

- Six (+ 2) predecessors in plain sight
  All except two are new
- All have (distinct) normal forms

Methods

- Two general methods (extend beyond numbers: trees, . . .)
- Several specific methods: one is particularly elegant
- Un-application

Outline

Koan: The Fundamental Tautology

$$\mathsf{c}_n \doteq \mathsf{c}_n\ \mathsf{succ}\ \mathsf{c}_0 \qquad (*)$$

Number representations

Equality

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Number representations

Definition; p_0 and supp are parameters

$$\mathsf{p}_n \;:\dot{=}\; \mathsf{c}_n\;\mathsf{supp}\;\mathsf{p}_0 \qquad n > 0 \qquad (**)$$

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In the sequence p_0, p_1, p_2,...

- p_0 is the initial element
- supp is the step: $\mathsf{supp}\;\mathsf{p}_n \rightsquigarrow^* \mathsf{p}_{(n+1)}$

Conversely, given p_0 and supp as parameters, (**) is the closed-form for the n-th element

General Recipe: Representation Change

By (**), transform c_n to a different number form p_n

If p_n are such that

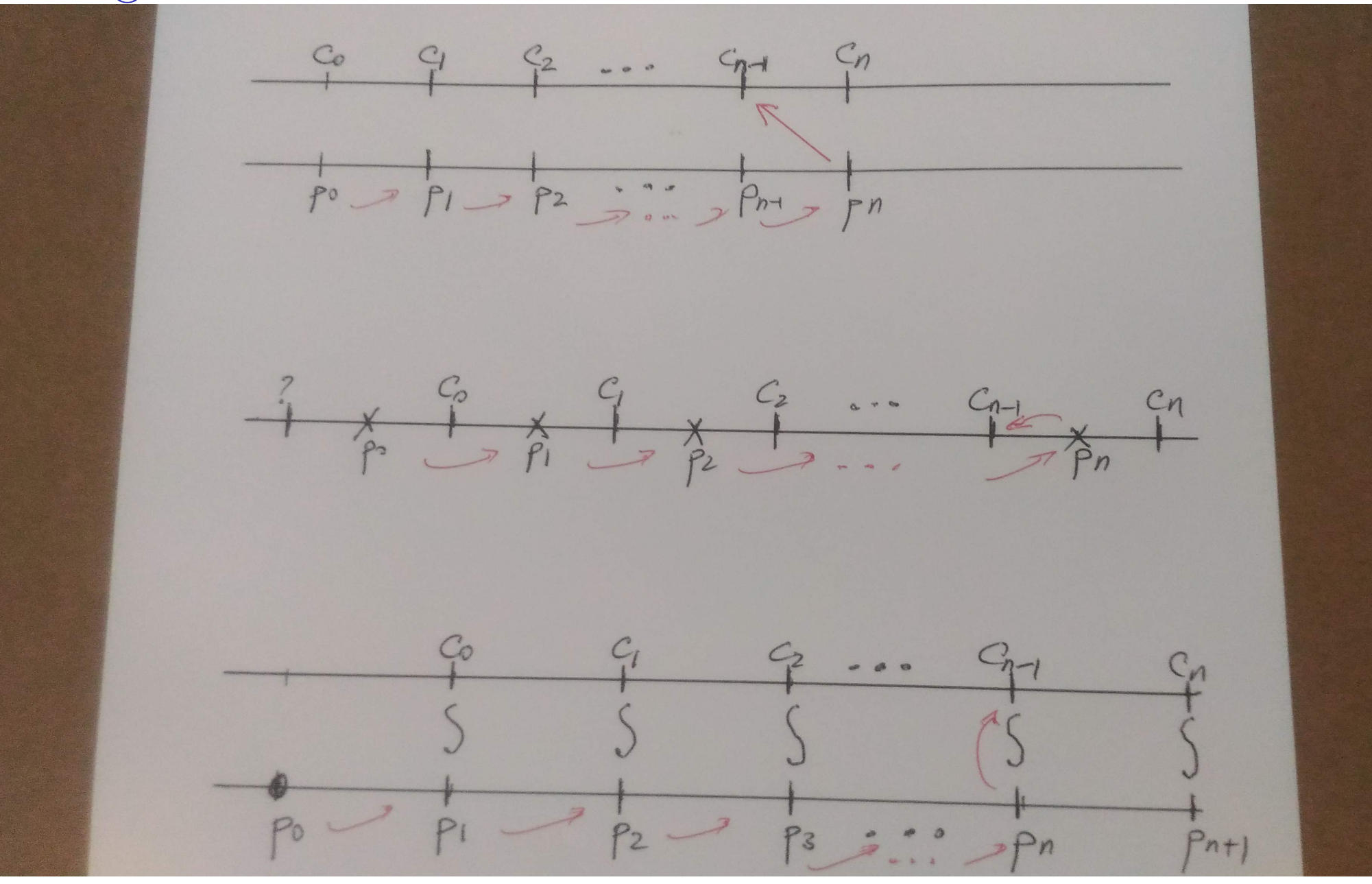
- $\mathsf{predp} : \mathsf{p}_{(n+1)} \mapsto \mathsf{p}_n$ and $\mathsf{proj} : \mathsf{p}_n \mapsto \mathsf{c}_n$ are easy to define, or
- $\mathsf{projpred} : \mathsf{p}_{(n+1)} \mapsto \mathsf{c}_n$ is easy to define

Then the predecessor is

$$\mathsf{pred} :\doteq \lambda n.\mathsf{projpred}\ \ (n\ \mathsf{supp}\ \mathsf{p}_0)$$

Now to find the fitting supp and p_0!

Diagram


c0 c1 c2 ... cn-1 cn
p0 p1 p2 ... pn-1 pn
? c0 c1 c2 ... cn-1 cn
p0 p1 p2 pn
c0 c1 c2 ... cn-1 cn
p0 p1 p2 p3 ... pn pn+1

Kleene Predecessor: midway numbers

Take p_n a point between two consecutive numbers c_n and c_{n-1}, represented as as pair $(\mathsf{c}_{n-1}\ ,\mathsf{c}_n)$:

$$\mathsf{p}_0 := (\mathsf{c}_{-1}\ ,\mathsf{c}_0) \qquad \mathsf{p}_1 := (\mathsf{c}_0\ ,\mathsf{c}_1) \qquad \mathsf{p}_2 := (\mathsf{c}_1\ ,\mathsf{c}_2) \qquad \ldots$$

The successor on p_n and $\mathsf{projpred}$

$$\begin{aligned} \mathsf{supp} &:\doteq \lambda p.(\mathsf{snd}\ p, \mathsf{succ}\ (\mathsf{snd}\ p)) \\ \mathsf{projpred} &:\doteq \qquad\qquad\qquad \mathsf{fst} \end{aligned}$$

Plugging into

$$\mathsf{pred} :\doteq \lambda n.\mathsf{projpred}\ \ (n\ \mathsf{supp}\ \mathsf{p}_0)$$

and normalizing gives the Kleene predecessor:

$$\lambda n.n\ (\lambda ps.s\ (p(\lambda xy.y))\ (\lambda fx.f\ (p(\lambda xy.y)f\ x)))\ (\lambda p.p\ (\lambda fx.x)\ (\lambda fx.x)$$

Even better General Recipe

Then the predecessor is

$$\mathsf{pred} :\doteq \lambda n.\, \mathsf{projpred}\ \ (n\ \mathsf{supp}\ \mathsf{p}_0)$$

where $\mathsf{projpred} : \mathsf{p}_{(n+1)} \mapsto \mathsf{c}_n$ and is easy to define

The simplest is to make $\mathsf{projpred}$ (a sort of) an identity!

$$\mathsf{p}_{(n+1)} \sim \mathsf{c}_n$$

Now to somehow find p_0, and supp to go with it...

Plan for the next

- $\sim$ is a bijection: general methods
- $\sim$ is the identity: specific methods

Outline

Shifted-by-one numbers

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read_int_opt : unit → int option

Adding an extra element to a set: X option

$\mathsf{p}_0 := \mathsf{None}$ $\quad$ $\mathsf{p}_1 := \mathsf{Some}\,\mathsf{c}_0$ $\quad$ $\mathsf{p}_2 := \mathsf{Some}\,\mathsf{c}_1$ $\quad$ $\ldots$

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And the predecessor is

$$\mathsf{pred} :\doteq \lambda n.\mathsf{fromSome}\ \ (n\ \mathsf{supp}\ \mathsf{None})$$

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$$\mathsf{pred} \doteq \lambda n.\mathsf{fromSome}\ \ (n\ \mathsf{supp}\ \mathsf{None})$$

$$\mathsf{None} := \lambda k.\lambda y.y \qquad\qquad \mathsf{Some} := \lambda x.\lambda k.\lambda y.kx$$

$$\mathsf{supp} \doteq \lambda p.\mathsf{Some}\ (p\ \mathsf{succ}\ \mathsf{c}_0)$$

The predecessor is

$$\lambda n.(n\ \mathsf{supp}\ \mathsf{p}_0)\ \mathsf{id}\ \mathsf{c}_0$$

or, in the desugared, normal form

$$\lambda n.n\ (\lambda pky.k\ (p(\lambda nfx.f\ (n\ f\ x))(\lambda fx.x)))\ (\lambda ky.y)\ (\lambda x.x)\ (\lambda fx.x)$$

Optimized Koan

$$\mathsf{c}_n \doteq \mathsf{c}_n\ \mathsf{succ}\ \mathsf{c}_0 \qquad (*)$$

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Shifted-by-one numbers, optimized

$$\mathsf{pred} \doteq \lambda n.\lambda f z.\mathsf{fromSome}_{fz}\ \ (n\ \mathsf{supp}_{fz}\ \mathsf{None}_{fz})$$

$$\mathsf{None}_{fz} := \lambda k.z \qquad\qquad \mathsf{Some}_{fz} := \lambda x.\lambda k.kx$$

$$\mathsf{supp}_{fz} \doteq \lambda p.\mathsf{Some}_{fz}\ (p\ f)$$

The predecessor is

$$\lambda n.\lambda f z.(n\ \mathsf{supp}_{fz}\ (\lambda k.z))\ \mathsf{id}$$

or, in the desugared, normal form

$$\lambda n f z.n\ (\lambda p k.k\ (p\ f))\ (\lambda k.z)\ (\lambda y.y)$$

Outline

Searching for -1

$$\mathsf{p}_{(n+1)} \equiv \mathsf{c}_n$$

Thus $\mathsf{p}_{(n+1)}$ are c_n themselves. What is p_0, a.k.a c_{-1}? The predecessor is

$$\mathsf{pred} :\doteq \lambda n.(n\ \mathsf{succ}'\ \mathsf{c}_{-1}) \qquad \text{where}$$

$$\mathsf{succ}' :\doteq \lambda n.\begin{cases} \mathsf{succ}\ n & \text{if } n \text{ is a Church numeral} \\ \mathsf{c}_0 & \text{if } n \text{ is } \mathsf{c}_{-1} \end{cases}$$

Now to find c_{-1} than can be distinguished from c_n

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$$\mathsf{c}_{-1} := \lambda f x.\mathsf{c}_0 \qquad\qquad \mathsf{succ}' :\doteq \lambda n.n\ \ \mathsf{id}\ (\mathsf{succ}\ n)$$

Thus the predecessor is

$$\lambda n.n\ (\lambda p.p\ (\lambda x.x)(\lambda f x.f\ (p\ f\ x)))\ (\lambda f x s z.z)$$

(found in the Summer of 1992)

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Or, in the desugared, normal form (size 25):

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succ° is a meta-circular successor:
raise the level, extend the domain

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Resolving my old puzzle

$$\mathsf{c}_n := \quad \lambda f z.\, f^{(n)}\, z$$

$$\begin{aligned} \mathsf{p}_{nz} &:= \quad \mathsf{Some}_z^{(n)}\ \mathsf{None}_z \qquad \text{where} \\ \mathsf{None}_z &:= \quad \lambda k.\, z \\ \mathsf{Some}_z &:= \quad \lambda a.\lambda k.\, k\ a \end{aligned}$$

Pattern-matching: un-application

$$\mathsf{p}_{nz}\ \mathsf{id} = \begin{cases} \mathsf{p}_{(n-1)z} & \text{if } n > 0 \\ z & \text{otherwise} \end{cases}$$

Reification	$\mathsf{c}_n \mapsto \mathsf{p}_{nz}$	$\mathsf{reif}_z := \lambda n.\, n\ \mathsf{Some}_z\ \mathsf{None}_z$
Reflection	$\mathsf{p}_{nz} \mapsto f^{(n)}\, z$	$\mathsf{refl}_f := \mathsf{fix}\ \lambda s.\lambda p.\, p\ (\lambda q.\, f\ (s\ q))$

Outline

Looking back

Algebras, with operations

c_0, p_0, c_{-1}	arity 0
succ, supp	arity 1

In functor form: $F(X) := 1 + X$

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is the unique homomorphism

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General approach, for any algebraic data type:

- write down the F functor
- apply either Böhm-Berarducci or Scott-Mogensen encoding
- *mechanically* obtain an efficient predecessor/extractor

Outline

What's else in the paper

- Sample generalization: left and right branch of a binary tree
- Outline of correctness proofs
  (algebraic/equational proofs)

Conclusions

Our reality may be very much like theirs. All this might just be an elaborate simulation running inside a little device sitting on someone's table.
StarTrek TNG, Episode 6x12, "Ship in a Bottle"